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Quantum sine-Gordon equation as the massive Thirring model*

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The sine-Gordon equation is the theory of a massless scalar field in one space and one time dimension with interaction density proportional to $\cos\beta\varphi$, where β is a real parameter. I show that if β^2 exceeds 8π, the energy density of the theory is unbounded below; if β^2 equals 4π, the theory is equivalent to the zero-charge sector of the theory of a free massive Fermi field; for other values of β, the theory is equivalent to the zero-charge sector of the massive Thirring model. The sine-Gordon soliton is identified with the fundamental fermion of the Thirring model.

I. INTRODUCTION

The sine-Gordon equation is the sophomoric but unfortunately standard name for the theory of a single scalar field in one space and one time dimension, with dynamics determined by the Lagrangian density[1]

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\alpha_0}{\beta^2}\cos\beta\varphi + \gamma_0\ . \tag{1.1}$$

Here the sum over repeated indices is implied, and α_0, β, and γ_0 are real parameters. Since we are free to redefine the sign of φ, we can always take β to be positive. Likewise, since we are free to shift φ by π/β, we can always take α_0 to be positive. We can gain some rough insight into the physical meaning of these parameters if, in the classical theory, we expand about the configuration of minimum energy ($\varphi = 0$):

$$\frac{\alpha_0}{\beta^2}\cos\beta\varphi = \frac{\alpha_0}{\beta^2} - \frac{\alpha_0}{2}\varphi^2 + \frac{\alpha_0\beta^2}{4!}\varphi^4 + \cdots . \tag{1.2}$$

From this we see that if we wish to adjust our zero of energy density so that the minimum energy is zero we must choose

$$\gamma_0 = -\frac{\alpha_0}{\beta^2}\ . \tag{1.3}$$

Also, α_0 is the "squared mass" (actually, since we are discussing a classical theory, squared inverse wavelength) associated with the spectrum of small oscillations about the minimum, and β is a parameter that measures the strength of the interactions between these small oscillations.

Of course, this is just a linearized analysis. The exact classical sine-Gordon equation has been extensively studied, and there exists an enormous literature concerning it.[2] Recently, some attempts have been made to use the known properties of the classical theory as a starting point for the investigation of the quantum theory.[3] This paper is a report on an attempt to investigate the properties of the quantum theory directly, without reference to the classical theory. The main results of the investigation are the following:

(1) As for any theory of a scalar field in two dimensions with nonderivative interactions, all divergences that occur in any order of perturbation theory can be removed by normal-ordering the Hamiltonian. For the sine-Gordon equation, this normal-ordering is equivalent to a multiplicative renormalization of α_0 and an additive renormalization of γ_0; β is not renormalized.[4]

(2) If β^2 exceeds 8π, the energy per unit volume is unbounded below, and the theory has no ground state.

(3) If β^2 is less than 8π, the theory is equivalent to the charge-zero sector of the massive Thirring model. This is a surprise, since the Thirring model is a canonical field theory whose Hamiltonian is expressed in terms of fundamental Fermi fields only.[5]

(4) In the special case $\beta^2 = 4\pi$, the sine-Gordon equation describes the charge-zero sector of a *free* massive Dirac field theory.

Further explanation of the last two points may be helpful to the reader.

The (massless) Thirring model[6] is a theory of a single Dirac field in one space and one time dimension, with dynamics determined by the Lagrangian density

$$\mathcal{L} = \bar{\psi} i\gamma_\mu\partial^\mu\psi - \tfrac{1}{2}g j^\mu j_\mu\ , \tag{1.4}$$

where

$$j^\mu = \bar{\psi}\gamma^\mu\psi\ , \tag{1.5}$$

and g is a free parameter, the coupling constant. The formal definition of j_μ as a product of Dirac fields is plagued with ambiguities, just as in four-dimensional theories, but, again just as in four-dimensional theories, these ambiguities can be resolved by demanding that j_μ obey the proper Ward identities.[7] Once this has been done, the Hamiltonian derived from Eq. (1.4) requires no further renormalizations. The model is exactly soluble and is sensible for g greater than minus π.

Within the massless Thirring model, it is possible to define a renormalized scalar density

$$\sigma = Z\bar{\psi}\psi \; , \tag{1.6}$$

where Z is a cutoff-dependent constant. The massive Thirring model is formally defined by adding a term proportional to σ to the Lagrangian density (1.4):

$$\mathcal{L} \to \mathcal{L} - m'\sigma \; . \tag{1.7}$$

Here m' is simply a real parameter; it is not to be identified with the mass of any presumed one-particle state. The massive Thirring model is not exactly soluble, and, to my knowledge, it is unknown whether (1.7) defines a physically sensible theory for any value of g. However, it is certainly true that every term in the perturbation series for the Green's functions of the theory in powers of m' is well defined, except for the trivial infrared problems associated with performing a mass perturbation expansion about a massless theory. These can be circumvented by a standard trick. Instead of (1.7), we consider the theory defined by

$$\mathcal{L} = \bar{\psi} i\gamma^{\mu}\partial_{\mu}\psi - \frac{g}{2} j^{\mu}j_{\mu} - m'\sigma f(x) \; , \tag{1.8}$$

where f is some function of space-time with compact support. The infrared divergences now disappear. A (probably unworkable) prescription for solving the theory would be to first sum up the perturbation series in m' with fixed f, and only then go to the limit $f=1$.

I can now explain more clearly results (3) and (4) above. The perturbation series described above for the massive Thirring model is term-by-term identical with a perturbation series (in α_0) for the sine-Gordon equation, if the following identifications are made between the two theories:

$$4\pi/\beta^2 = 1 + g/\pi \; , \tag{1.9}$$

$$-\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_{\nu}\varphi = j^{\mu} \; , \tag{1.10}$$

$$\frac{\alpha_0}{\beta^2}\cos\beta\varphi = -m'\sigma \; . \tag{1.11}$$

Of course, this analysis does not show that either the massive Thirring model or the sine-Gordon equation exists, in the strict sense of constructive field theory. I believe it is fair to say, though, that it does show that if either theory exists, it is equivalent to the other. Please note that Eq. (1.9) implies that if $\beta^2 = 4\pi$, then $g=0$. This is result (4).

The analysis that leads to these results is explained in Secs. II, III, and IV below. Section V contains some remarks about the meaning of the results, and some conjectures.

II. THE HAMILTONIAN

For any scalar field theory in two dimensions with nonderivative interactions, the only ultraviolet divergences that occur in any order of perturbation theory come from graphs that contain a closed loop consisting of a single internal line, that is to say, graphs in which two fields at the same vertex are contracted with each other. Thus, for example, the graph of Fig. 1(a) is ultraviolet-divergent, while that of Fig. 1(b) is not. Thus, all ultraviolet divergences can be removed by normal-ordering the interaction Hamiltonian in the interaction picture.

However (at least for the moment), we are not interested in the interaction picture but in the Schrödinger picture. It is easy to see what the normal-ordering prescription corresponds to in this picture. Schrödinger-picture operators are given as functions of the field $\varphi(x)$ and the canonical momentum density $\pi(x)$, where x is the spatial coordinate. If we define operators $a(k,m)$ by

$$\varphi(x) = \int \frac{dk}{2\pi}\left[\frac{1}{2\omega(k,m)}\right]^{1/2}\left[a(k,m)e^{-ikx} + a^{\dagger}(k,m)e^{ikx}\right] \; , \tag{2.1}$$

and

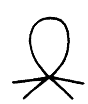
$$\pi(x) = i\int \frac{dk}{2\pi}\left[\frac{\omega(k,m)}{2}\right]^{1/2}\left[a(k,m)e^{-ikx} - a^{\dagger}(k,m)e^{ikx}\right] \; , \tag{2.2}$$

where

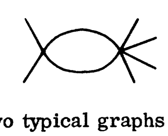
$$\omega(k,m) = (k^2 + m^2)^{1/2} \; , \tag{2.3}$$

then the normal-ordered Schrödinger operator corresponds to the operator rearranged with all the a's on the right and all the $a^{\dagger}$'s on the left.

However, this prescription is ambiguous, because it does not tell us what m is. Of course, if we are really doing interaction-picture perturbation theory, it would be senseless to choose m to be other than that mass which occurs in the



(a)



(b)

FIG. 1. Two typical graphs in a scalar field theory in two dimensions with nonderivative interactions; (a) is logarithmically divergent; (b) is convergent.

free Hamiltonian. However, for the sine-Gordon equation, it is not clear what is the most profitable way to divide the Hamiltonian into a free and an interaction part, or, indeed, whether any such division is profitable. Therefore, for the moment at least, we will not specify m, and we will denote by N_m the normal-ordering operation defined by the mass m.

[This is off the main line of the argument, but it is amusing to consider what would happen, in perturbation theory, if we chose m to be different from μ, the mass in the free Hamiltonian. In this case, it is easy to see that the divergent loop integral of Fig. 1(a) is replaced by a convergent integral according to the following prescription:

$$\int \frac{d^2k}{k^2+\mu^2} \to \int d^2k\left(\frac{1}{k^2+\mu^2}-\frac{1}{k^2+m^2}\right) . \tag{2.4}$$

Note that if $m=\mu$ (conventional normal-ordering), the graph is canceled completely (the usual result).]

So much for generalities. We now turn to the sine-Gordon Hamiltonian density,

$$\mathcal{H} = -\tfrac{1}{2}\pi^2+\tfrac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 - \frac{\alpha_0}{\beta^2}\cos\beta\varphi-\gamma_0 . \tag{2.5}$$

Let us begin with the cosine term. Wick's theorem tells us that, for a free field of mass m, and for any space-time function $J(x)$,

$$\exp\left[i\int J(x)\varphi(x)d^2x\right]=N_m\exp\left[i\int J(x)\varphi(x)d^2x\right]\exp\left[-\tfrac{1}{2}\int J(x)\Delta(x-y;m)J(y)d^2x\,d^2y\right] , \tag{2.6}$$

where Δ is the free-field two-point Wightman function. [If we replace the unordered exponential in Eq. (2.5) by a time-ordered exponential, the Wightman function is replaced by the Feynman propagator.] For small spacelike separation

$$\Delta(x;m) = -\frac{1}{4\pi}\ln cm^2x^2+O(x^2) , \tag{2.7}$$

where c is a numerical constant (related to Euler's constant) and

$$x^2=-x_\mu x^\mu , \tag{2.8}$$

and is positive for spacelike separation. We cut off the theory by replacing $\Delta(x;m)$ by

$$\Delta(x;m)-\Delta(x;\Lambda)\equiv\Delta(x;m;\Lambda) , \tag{2.9}$$

where Λ is a large mass, the cutoff. This is nonsingular at the origin,

$$\Delta(0;m;\Lambda) = -\frac{1}{4\pi}\ln\frac{m^2}{\Lambda^2} . \tag{2.10}$$

We can now use Eq. (2.6) for J a δ function. We find

$$e^{i\beta\varphi}=N_m\left(\frac{m^2}{\Lambda^2}\right)^{\beta^2/8\pi}e^{i\beta\varphi} . \tag{2.11}$$

This has been derived in the interaction picture, but must also be true in the Schrödinger picture, since it only involves fields at the same time. Thus, if we define, for some arbitrarily chosen mass m,

$$\alpha=\alpha_0\left(\frac{m^2}{\Lambda^2}\right)^{\beta^2/8\pi} , \tag{2.12}$$

then

$$\alpha_0\cos\beta\varphi=\alpha N_m\cos\beta\varphi . \tag{2.13}$$

Of course, if we had chosen a different mass (call it μ) to define the theory, this would just have led to a finite multiplicative redefinition of α. Indeed, from Eq. (2.11),

$$N_m\cos\beta\varphi=\left(\frac{\mu^2}{m^2}\right)^{\beta^2/8\pi}N_\mu\cos\beta\varphi . \tag{2.14}$$

This identity will be very important to us shortly.

Much less care needs to be taken in normal-ordering the remainder of the Hamiltonian density, since this is just a quadratic form in the fundamental fields, and therefore the only effect of normal-ordering is to add a constant. For brevity of subsequent notation, we define

$$\mathcal{H}_0=\tfrac{1}{2}\pi^2+\tfrac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 . \tag{2.15}$$

Then, from Eqs. (2.1) and (2.2),

$$\mathcal{H}_0=N_m\mathcal{H}_0+E_0(m) , \tag{2.16}$$

where

$$E_0(m)=\int\frac{dk}{8\pi}\,\frac{2k^2+m^2}{\omega(k,m)} . \tag{2.17}$$

Thus, if we define

$$\gamma=\gamma_0+E_0(m) , \tag{2.18}$$

then

$$\mathcal{H}_0-\gamma_0=N_m\mathcal{H}_0-\gamma . \tag{2.19}$$

Parallel to Eq. (2.14) we have

$$\begin{aligned}N_m\mathcal{H}_0&=N_\mu\mathcal{H}_0+E_0(\mu)-E_0(m)\\&=N_\mu\mathcal{H}_0+\frac{1}{8\pi}(\mu^2-m^2) .\end{aligned} \tag{2.20}$$

Assembling all this, we find the cutoff-independent form of the Hamiltonian density:

$$\mathcal{H} = N_m \left(\mathcal{H}_0 - \frac{\alpha}{\beta^2}\cos\beta\varphi - \gamma\right) . \tag{2.21}$$

where α, β, and γ are finite parameters. This is result (1) of the Introduction.

III. A VARIATIONAL COMPUTATION

Considerable insight into the physical import of the reordering equations, (2.14) and (2.20), can be obtained by considering the following trivial problem. Suppose we are given the Hamiltonian density

$$\mathcal{H} = N_m(\mathcal{H}_0 + \tfrac{1}{2}\beta^2\varphi^2) . \tag{3.1}$$

This is a free-field Hamiltonian density, normal-ordered with a perverse mass; that is to say, m is not necessarily equal to β.

Let us attempt to find the ground state of this theory by the Rayleigh-Ritz variational method. For our trial states we will use the vacuum states appropriate to a free field of mass μ. These states are defined by

$$a(k, \mu)|0, \mu\rangle = 0 . \tag{3.2}$$

The computation is made trivial by reordering. Expanding Eq. (2.14) in powers of β^2, we find

$$N_m(\tfrac{1}{2}\beta^2\varphi^2) = N_\mu(\tfrac{1}{2}\beta^2\varphi^2) - \frac{\beta^2}{8\pi}\ln\frac{\mu^2}{m^2} . \tag{3.3}$$

Thus, from this and Eq. (2.20),

$$\mathcal{H} = N_\mu(\mathcal{H}_0 + \tfrac{1}{2}\beta^2\varphi^2) + \frac{1}{8\pi}\left(\mu^2 - m^2 - \beta^2\ln\frac{\mu^2}{m^2}\right) , \tag{3.4}$$

whence

$$\langle 0, \mu|\mathcal{H}|0, \mu\rangle = \frac{1}{8\pi}\left(\mu^2 - m^2 - \beta^2\ln\frac{\mu^2}{m^2}\right) . \tag{3.5}$$

As a function of μ, the right-hand side of this equation assumes its minimum value when μ is β. This is, of course, the correct result.

Now let us perform exactly the same computation for the sine-Gordon equation:

$$\mathcal{H} = N_m \left(\mathcal{H}_0 - \frac{\alpha}{\beta^2}\cos\beta\varphi - \gamma\right) . \tag{2.21}$$

From Eqs. (2.14) and (2.20) we find

$$\mathcal{H} = N_\mu\left[\mathcal{H}_0 - \frac{\alpha}{\beta^2}\left(\frac{\mu^2}{m^2}\right)^{\beta^2/8\pi}\cos\beta\varphi + \frac{1}{8\pi}(\mu^2 - m^2) - \gamma\right] , \tag{3.6}$$

whence

$$\langle 0, \mu|\mathcal{H}|0, \mu\rangle = -\frac{\alpha}{\beta^2}\left(\frac{\mu^2}{m^2}\right)^{\beta^2/8\pi} + \frac{1}{8\pi}(\mu^2 - m^2) - \gamma . \tag{3.7}$$

In contrast to Eq. (3.5), the right-hand side of this equation is unbounded below, as μ goes to infinity, if β^2 exceeds 8π. Thus, in this case, the energy of the theory (even if we restrict it to a box of finite volume) is unbounded below. The theory has no ground state, and is physically nonsensical. This is result (2) of the Introduction.

IV. PERTURBATION THEORY

Following the plan set forth in Sec. I, we will now construct a perturbation series (in α) for the Green's functions of the quantum sine-Gordon equation. The Green's functions we will choose to compute will be those corresponding to the vacuum expectation value of the time-ordered product of a string of interaction Hamiltonian densities. The interaction density is a somewhat unconventional choice for the local field that characterizes the theory, but, in principle, there is no reason to prefer any local field to any other for this purpose, and by choosing the interaction density we will be able to shorten our labors significantly.

Before beginning the computation, we will have to straighten out the problems associated with the infrared divergences which plague all perturbative calculations about a massless field theory. These problems are notoriously acute for a massless scalar field theory in two dimensions. For example, the very expression for the field in terms of annihilation and creation operators, Eq. (2.2), is infrared-divergent; as a consequence of this, it is not possible to define $\varphi(x)$ as a Wightman field in the conventional sense.[8] (This does not mean that the theory of a free massless scalar meson in two dimensions is nonsense. For example, $\partial_\mu\varphi$ is perfectly well defined—the extra power of momentum brought in by differentiation eliminates the infrared divergence—and can be used as the Wightman field.)

The simplest way to circumvent these problems is to give the scalar meson a small mass μ. This will enable us to compute the terms in the perturbation expansion by conventional techniques (e.g., Wick's theorem). We will then send μ to zero. Of course, this will still leave us with the usual infrared problems, just like those which arise for the Thirring model, discussed in Sec. I. We will handle these by the method described previously, i.e., by multiplying the interaction by a space-time function of compact support, $f(x)$.

Thus, we are led to the Hamiltonian density

$$\mathcal{H} = N_m\left[\mathcal{H}_0 + \tfrac{1}{2}\mu^2\varphi^2 + f(x)\left(-\frac{\alpha}{\beta^2}\cos\beta\varphi - \gamma\right)\right] . \tag{4.1}$$

To find the desired Green's functions we must (1) compute the vacuum persistence amplitude to all orders in perturbation theory; (2) send μ to zero, keeping f fixed; (3) sum up the series; (4) send f to one; and (5) compute appropriate variational derivatives with respect to f about $f=1$. The last three steps are beyond my analytic ability; fortunately, they are not necessary for our purposes, for we will establish the identity with the perturbation series (in m') for the massive Thirring model at the end of step (2).

It is instructive to compute a slightly more general object than is needed for our immediate purposes. Consider

$$T\langle 0,\mu|\prod_i N_m e^{i\beta_i\varphi(x_i)}|0,\mu\rangle \ , \tag{4.2}$$

where φ is a free field of mass μ, the β's are real parameters, and the x's are space-time points (within the support of f). Because the x's are restricted to a finite region, we can uniformly apply the short-distance approximation for the Feynman propagator. This is identical to Eq. (2.7), except for factors of $i\epsilon$, which I will suppress anyway, for notational simplicity:

$$\Delta_F(x) = -\frac{1}{4\pi}\ln c\mu^2 x^2 \ . \tag{4.3}$$

If we use the identity

$$N_m e^{i\beta\varphi} = \left(\frac{\mu^2}{m^2}\right)^{\beta^2/8\pi} N_\mu e^{i\beta\varphi} \ , \tag{4.4}$$

we can compute (4.2) directly, using Wick's theorem, Eq. (2.6), and ignoring all terms involving contractions of fields at the same point. The result is easily seen to be

$$\left(\frac{\mu^2}{m^2}\right)^{(\Sigma\beta_i{}^2/8\pi)} \prod_{i>j}\,[c\mu^2(x_i-x_j)^2]^{\beta_i\beta_j/4\pi} \ . \tag{4.5}$$

The critical point is that this expression is proportional to

$$\mu^{(\Sigma\beta_i)^2/4\pi} \ . \tag{4.6}$$

Thus, if the β's sum to zero, (4.5) is independent of μ; if they do not sum to zero, (4.5) vanishes as μ goes to zero. This result is an important consistency check on our computation, for it is just what we should expect in a massless free scalar field theory in two dimensions. The Lagrangian for such a theory is invariant under the transformation

$$\varphi \to \varphi + \lambda \ , \tag{4.7}$$

where λ is a real parameter. Formally, this implies

$$e^{i\beta\varphi} \to e^{i\beta\lambda} e^{i\beta\varphi} \ , \tag{4.8}$$

which in turn implies that the vacuum expectation values vanish unless the sum of the β's is zero—exactly what we have found. This argument breaks down in greater than two dimensions because the symmetry (4.7) is spontaneously broken. However, in two dimensions there is no spontaneous breakdown of continuous symmetries,[9] so the argument should be valid—and it is.

Now let us specialize these results to the sine-Gordon equation. The object of interest is

$$\frac{\alpha}{\beta^2} N_m \cos\beta\varphi = \frac{\alpha}{2\beta^2}(A_+ + A_-) \ , \tag{4.9}$$

where

$$A_\pm = N_m e^{\pm i\beta\varphi} \ . \tag{4.10}$$

In the limit of vanishing μ, the only nonzero terms in the perturbation expansion are those with equal numbers of A_+'s and A_-'s. These terms are given by

$$T\langle 0|\prod_{i=1}^{n} A_+(x_i)A_-(y_i)|0\rangle = \frac{\prod_{i>j}[(x_i-x_j)^2(y_i-y_j)^2c^2m^4]^{\beta^2/4\pi}}{\prod_{i,j}[cm^2(x_i-y_j)^2]^{\beta^2/4\pi}} \ . \tag{4.11}$$

Now let us construct the corresponding perturbation series (in m) for the massive Thirring model. Here the object of interest is

$$m'\sigma = m'(\sigma_+ + \sigma_-) \ , \tag{4.12}$$

where

$$\sigma_\pm = \tfrac{1}{2}Z\bar\psi(1\pm\gamma_5)\psi \ , \tag{4.13}$$

and Z is the (cutoff-dependent) multiplicative renormalization constant referred to in Sec. I. Because the massless Thirring model is chirally invariant, the only nonzero terms in the perturbation expansion are those with equal number of σ_+'s and σ_-'s. These are best calculated in a basis in which

$$\gamma_0 = \sigma_x, \quad \gamma_1 = i\sigma_y, \quad \gamma_5 = -\sigma_z \ . \tag{4.14}$$

In this basis

$$\sigma_+ = Z\psi_1{}^\dagger\psi_2, \quad \sigma_- = Z\psi_2{}^\dagger\psi_1 \ , \tag{4.15}$$

where the subscripts indicate the components of the Dirac spinor.

The expressions we need for the computation can be found in the beautiful paper of Klaiber,[6] which contains closed forms for the vacuum expectation values of arbitrary strings of ψ_i's and $\psi_i{}^\dagger$'s. Klaiber's formulas are lengthy, and I will not reproduce them here, but merely state the consequences of them that are relevant to our purposes. All of these can be obtained from Klaiber's work by trivial manipulations; for the skeptical reader who wishes to verify that I have made no algebraic errors, I will adhere as closely as possible to Klaiber's notation.

In an arbitrary Green's function, if we bring a $\psi_1^\dagger$ and a ψ_2 to the same space-time point, the Green's function becomes singular. However, if we define,

$$\sigma_+(x) \propto \lim_{y\to x} (x-y)^{2\delta}\psi_1^\dagger(x)\psi_2(y) , \tag{4.16}$$

then σ_+ has finite matrix elements. Here Klaiber's parameter δ is given by

$$\delta = \frac{g}{4\pi}\,\frac{2\pi+g}{\pi+g} , \tag{4.17}$$

where g is the coupling constant as defined in Sec. I, and the proportionality sign is to indicate that σ_+ is defined only up to a finite multiplicative renormalization. The local field σ_- is defined by the adjoint equation.

With this definition, it follows from Klaiber's formulas that

$$\langle 0|T\prod_{i=1}^{n} \sigma_+(x_i)\sigma_-(y_i)|0\rangle = (\tfrac{1}{2})^{2n}\frac{\prod_{i>j}[(x_i-x_j)^2(y_i-y_j)^2M^4]^{(1+b/\pi)}}{\prod_{i,j}[M^2(x_i-y_j)^2]^{(1+b/\pi)}} , \tag{4.18}$$

where M is an arbitrary mass, the reflection of the arbitrary finite renormalization in the definition of the σ's, and Klaiber's parameter b is given by

$$b = -\frac{g}{1+g/\pi} . \tag{4.19}$$

This equation is identical with Eq. (4.7) if we make the identifications

$$-\sigma_\pm = \tfrac{1}{2}A_\pm , \tag{4.20}$$

$$M^2 = cm^2 , \tag{4.21}$$

and

$$\frac{1}{1+g/\pi} = \frac{\beta^2}{4\pi} . \tag{4.22}$$

Thus, the two perturbation theories are identical if we choose

$$m' = \alpha/\beta^2 , \tag{4.23}$$

whence

$$-m'\sigma = \frac{\alpha}{\beta^2}N_m\cos\beta\varphi . \tag{4.24}$$

Equations (4.22) and (4.24) are Eqs. (1.9) and (1.11) of the introduction. These equations do not depend on properly matching the renormalization conventions in the two theories, Eq. (4.21). In contrast, Eqs. (4.20) and (4.23) do depend on proper matching, and therefore have no convention-independent meaning.

Now let us turn to the current. In the Thirring model, Klaiber has shown that

$$[j^\mu(x), \psi(y)] = -[g^{\mu\nu} + (1+g/\pi)^{-1}\epsilon^{\mu\nu}\gamma_5] \times\psi(y)D_\nu(x-y) , \tag{4.25}$$

where D_ν is the gradient of the massless free scalar commutator. At equal times,

$$D_\nu(0, x^1) = g_{\nu 0}\delta(x^1) . \tag{4.26}$$

Thus, the current is properly normalized; that is to say, if we identify the conserved charge associated with the current as fermion number, the field carries fermion number minus one. From Eq. (4.2r) and the definition of the σ's, Eq. (4.12), it follows that

$$[j^\mu(x), \sigma_\pm(y)] = \mp 2\left(1+\frac{g}{\pi}\right)^{-1}\epsilon^{\mu\nu}D_\nu(x-y)\sigma_\pm(y) . \tag{4.27}$$

This equation completely defines j^μ in the sector in which we are interested, the set of all states that can be created from the vacuum by application of the σ's in the massless Thirring model.

In the massless scalar theory defined in the first part of this section, $\partial_\nu\varphi$ is a free field. Thus, it is trivial to compute its commutators with the A's; we find

$$[\partial_\nu\varphi(x), A_\pm(y)] = \pm\beta D_\nu(x-y)A_\pm(y) . \tag{4.28}$$

Since the A's are identified with the σ's, it follows that

$$j^\mu = -\frac{2}{\beta(1+g/\pi)}\epsilon^{\mu\nu}\partial_\nu\varphi = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\varphi . \tag{4.29}$$

This is the last of the results announced in the Introduction and completes the argument.

V. PUZZLING QUESTIONS, BUT NOT BEYOND ALL CONJECTURE

A. Bounds on coupling constants

We have seen that the sine-Gordon equation is physically nonsensical if β^2 exceeds 8π. I have not been able to show that it is physically sensible for any β^2 less than 8π, except for two points: zero, where the theory is a free massive Bose field theory, and 4π, where the theory is a free massive Fermi field theory. The natural conjecture is that the theory is sensible for β^2 less than some upper bound lying somewhere between 4π and 8π, but I know of no evidence to support this conjecture.

In the language of the Thirring model, $\beta^2 < 8\pi$ corresponds to

$$g > -\pi/2 . \tag{5.1}$$

From the viewpoint of the Thirring model, this is a surprising result, for this bound is stronger than the corresponding one for the massless model, $g > -\pi$. Thus, there is a whole range of coupling constants for which the massless Thirring model is perfectly sensible, but for which the addition of a mass term, no matter how small its coefficient, causes the bottom to drop out of the energy spectrum. Is it possible to understand this phenomenon purely in Thirring-model language? Yes, it is; here is the argument:

The massless Thirring model is exactly scale-invariant; there is a conserved scale current and a unitary dilatation operator. This implies that the Hamiltonian density of the massless model must transform under dilatations according to

$$\lambda:\ \mathcal{H}^{(0)}(x) \to \lambda^2 \mathcal{H}^{(0)}(\lambda x) , \tag{5.2}$$

where λ is the dilatation parameter. As can be seen from Eq. (4.16), the field σ has anomalous dimension b/π; thus,

$$\lambda:\ \sigma(x) \to \lambda^{1/(1+g/\pi)} \sigma(\lambda x) . \tag{5.3}$$

Now let us imagine performing a variational estimate of the ground-state energy of the massive model, using as our trial states a family of translation-invariant states which are dilatation transforms of each other. If we label these states by the dilatation parameter λ, then

$$\langle \lambda | \mathcal{H} | \lambda \rangle = \lambda^2 \langle 1 | \mathcal{H}^{(0)} | 1 \rangle + m' \lambda^{1/(1+g/\pi)} \langle 1 | \sigma | 1 \rangle . \tag{5.4}$$

If g is less than $-\pi/2$, the second term dominates as λ goes to infinity. Thus, we are in trouble if the second term is negative. But since σ can be turned into $-\sigma$ by a chiral rotation, we can always find a state such that the expectation value of σ is negative. Q.E.D.

B. Metamorphosis of fermions into bosons

One of the most striking features of the results established here is that a theory which is "obviously" a theory of fermions is equivalent to a theory which is "obviously" a theory of bosons.[10] This peculiar result should seem even more peculiar to you if you have followed the proof of it in Sec. IV, for the proof rested on identifying a certain sector of the massless Thirring model with a certain sector of massless free scalar theory. That *these* theories are equivalent in any sense seems too preposterous to believe. Is there no way of telling fermions from bosons? The answer is no, there is not—if one can measure only a restricted set of local fields, *and* if the particles are massless, *and* if the world is two-dimensional.

This can most easily be seen by a simpler and less singular theory than either of the two at hand: the theory of a free massless Dirac field. Let us suppose that, in such a theory, we can only measure the Green's functions associated with local operators of charge zero. That is to say, we can only make particle-antiparticle pairs out of the vacuum, not single particles. This is not an unrealistic restriction; it is precisely that restriction which we face in the real four-dimensional world, where we can only measure charge-zero observables. Nevertheless, in the real world, we have no difficulty in determining that electrons are fermions and charged pions are bosons. The reason is that after we create a pair in the real world, the components of the pair separate; thus, if we wait long enough, we can arrange matters so one of the components of the pair is in our laboratory and the other is at the orbit of Pluto and traveling outward. However, for massless particles in two dimensions, it is quite possible to make a pair that never separates. Such a pair consists of two particles moving in the same direction. The wave functions do not spread; they just move on steadily at the speed of light, and the particles never get away from each other. If the particles had a mass, or if the world were of greater than two dimensions, this would not be possible.

Another way of seeing the same thing is to study the charge-zero sector of the Fock space of the massless Dirac theory. This contains states corresponding to a fermion and antifermion, each in a normalizable state with spatial momentum support restricted to the positive axis. Even though this is a normalizable state, it is still an eigenstate of $P_\mu P^\mu$, with eigenvalue zero; all of the two-momenta in the supports of the individual particle states are aligned null vectors. Thus, if we adopt the usual definition of a particle, a normalizable eigenstate of P^2, we must say that the Fock space of a massless Dirac field contains massless Bose particles. Again, this is only possible for mass zero and dimension two.

However, all this pathology disappears, even in two dimensions, if the particles are massive. Then pairs will separate, and there are no two-particle normalizable mass eigenstates. Thus, the problems of interpretation discussed above are not really relevant to the theories considered here, theories of massive particles. Here the same kind of experiments that tell us unambiguously in four dimensions that electrons are fermions should tell us unambiguously that, for example, the only particles in the sine-Gordon equation with $\beta^2=4\pi$ are free massive fermions.[11]

C. Classical theory, quantum solitons, and a conjecture

To obtain insight into the connection between the quantum sine-Gordon equation and its classical limit, let us introduce the rescaled variables

$$\varphi'=\beta\varphi, \quad \gamma_0'=\gamma_0/\beta^2 . \tag{5.5}$$

In terms of these variables, the Lagrangian density is

$$\mathcal{L}=\frac{1}{\beta^2}\left[\tfrac{1}{2}(\partial_\mu\varphi')^2+\alpha_0\cos\varphi'+\gamma_0'\right] . \tag{5.6}$$

Since, in classical physics, multiplying the Lagrangian density by a constant has no effect on the physics (other than trivially redefining the scale of energy), β is an irrelevant parameter in the classical sine-Gordon equation. In the quantum theory, as we have seen, this is not so; this is because the relevant object for quantum physics is $\mathcal{L}/\hbar$, and rescaling the Lagrangian is equivalent to rescaling $\hbar$. (We have obscured this until now by choosing our units so $\hbar=1$.) Thus, in the quantum theory, the relevant parameter is $\beta^2\hbar$. Therefore, semiclassical approximation schemes, such as the expansion of Goldstone and Jackiw,[3] or the modified WKB approximation of Dashen, Hasslacher, and Neveu,[3] being small-$\hbar$ approximations, are necessarily also small-β approximations.[12] Since small β is large g, these methods give us information about the Thirring model for *large* coupling constants. What is this information?

The classical equation possesses a time-independent solution of finite energy, the famous soliton

$$\varphi'(x)=4\tan^{-1}\exp(x\sqrt{\alpha_0})\equiv f(x) . \tag{5.7}$$

Of course, spatial translations and Lorentz transformations of this solution are also solutions. The semiclassical analyses show that these solutions correspond to a particle in the quantum theory, the quantum soliton. In the leading approximation, the mass of the quantum soliton is proportional to $1/\beta^2$, and the states of the quantum soliton correspond to certain superpositions of coherent states of the φ field, such that for a one-soliton state approximately localized at the point a,

$$\langle\varphi'(x)\rangle=f(x-a) . \tag{5.8}$$

All one-soliton states are orthogonal to states made by applying local polynomials in φ to the vacuum state. [This is because all such states have the property that the expectation value of φ' goes to zero as x goes to plus or minus infinity, while (5.8) goes to zero at minus infinity but goes to 2π for plus infinity.] However, states containing equal numbers of solitons and antisolitons do not have this property. For a state consisting of a widely separated soliton-antisoliton pair

$$\langle\varphi'(x)\rangle\approx f(x-a)-f(x+a) , \tag{5.9}$$

where the approximation sign indicates that, even in the leading semiclassical approximation, this formula is valid only for large a. It is not known whether the quantum soliton is a fermion or a boson.

(This last remark probably requires amplification. The spin-statistics theorem is useless, because there is no spin in two dimensions, since there is no little group. Of course, even in two dimensions there is a connection between the statistics of a particle and the Lorentz-transformation properties of the local field that creates the particle from the vacuum, but since the soliton is not created by a local field, this is also of no help. The semiclassical methods are in principle capable of answering the question (for example, by studying the symmetry of soliton-soliton scattering), but the necessary computations have not yet been done.)

I conjecture that the quantum soliton is a fermion, and is, in fact, the fundamental fermion of the massive Thirring model. Here are my reasons:

(1) A similar time-independent solution of finite energy exists for classical two-dimensional φ^4 theory. Goldstone and Jackiw[3] have shown that this particle is a fermion. Unfortunately, their argument depends on the identification of particle and antiparticle, possible for φ^4 but not for the sine-Gordon equation.

(2) Equation (1.10) is an exact result. If we apply it to a widely separated soliton-antisoliton pair, we find

$$\begin{aligned}\langle j_0(x)\rangle&=\frac{1}{2\pi}\langle\partial_1\varphi'(x)\rangle\\&=\frac{1}{2\pi}\frac{d}{dx}[f(x-a)-f(x+a)] .\end{aligned} \tag{5.10}$$

The right-hand side of this expression is peaked about the points $-a$ and a, the locations of the soliton and antisoliton. The integrated charge over the soliton peak is $+1$; that over the antisoliton peak is -1. Since the charge in question is fermion number for the Thirring model, this is evidence for the conjecture.

(3) In the massive Thirring model, we have defined the coupling constant such that positive g

corresponds to an attractive force between fermions and antifermions. In the nonrelativistic limit, the attractive interaction is a δ-function potential; in one spatial dimension, such a potential produces a bound state for arbitrarily small g. As we increase g, we would expect the mass of this bound state to decrease, becoming as small as possible (zero) as g goes to infinity. This bound state is the obvious candidate for the fundamental meson of the sine-Gordon equation.

Thus, I am led to conjecture a form of duality, or nuclear democracy in the sense of Chew, for this two-dimensional theory. A single theory has two equally valid descriptions in terms of Lagrangian field theory: the massive Thirring model and the quantum sine-Gordon equation. The particles which are fundamental in one description are composite in the other: In the Thirring model, the fermion is fundamental and the boson a fermion-antifermion bound state; in the sine-Gordon equation, the boson is fundamental and the fermion a coherent bound state. In the Thirring-model description, as the coupling constant goes to infinity, the bound-state mass goes to zero; in the sine-Gordon description, as the coupling constant vanishes, the coherent-bound-state mass goes to infinity; these are just two ways of describing the behavior of a single mass ratio in a single limit.

Speculation on extending these ideas to four dimensions is left as an excercise for the reader.

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APPENDIX (ADDED IN PROOF)

Since this paper was accepted for publication, there have been several new developments:

1. I have learned from Schroer[13] that he has independently obtained many of the results in this paper. Schroer has also pointed out that many of the results obtained here are in close correspondence with the results of the studies of one-dimensional electron gasses by Luther and collaborators.[14] Luther and I are in total agreement with Schroer on this point; we are also united in our embarrassment that we were incapable of reaching this conclusion unprompted. (Our offices are on the same corridor.)

2. Mandelstam[15] has been able to construct the Fermi field operators as (nonlocal) functions of the canonical Bose field, following the methods of Dell'Antonio *et al.*[16]

3. Fröhlich[17] has been able to rigorously prove that the sine-Gordon Hamiltonian defines a physically sensible theory for $\beta^2 < 16/\pi$. (The restriction on β is connected only with certain technical details of the proof, not with any obvious pathology of the physics.) He has also been able to establish similar results for the sine-Gordon Hamiltonian with a mass term (the massive Thirring-Schwinger model).

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[1]Notation: $g_{00}=-g_{11}=1$, $\epsilon_{01}=-\epsilon_{10}=1$. The symbol x is used for both a space-time point and for the associated spatial coordinate x^1; which is meant should always be clear from the context.

[2]For a review, see A. Scott, F. Chu, and D. McLaughlin, Proc. IEEE 61, 1443 (1973).

[3]J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975); R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974); 10, 4130 (1974); 10, 4138 (1974); K. Cahill, Ecole Polytechnique report (unpublished).

[4]This result is trivial and is well known to workers in the field.

[5]I thank Andre Neveu for pointing out to me that this result is prefigured in two papers by T. H. R. Skyrme [Proc. R. Soc. A247, 260 (1958); A262, 237 (1961)]. Skyrme argued that the soliton modes of the sine-Gordon equation were fermions, and that the interaction between the fermions was of Thirring-model type. This is in agreement with my results; however, others of Skyrme's conclusions contradict mine: He argued that the identification with fermions could only be made for $\beta^2=4\pi$ (this is Skyrme's condition $\epsilon=\frac{1}{2}\hbar c$), he found that a fermion interaction existed in this case, and he identified the fermion current with a very different operator than I do. I regret that I do not understand Skyrme's methods well enough to know whether an improved version of them, taking more careful account of renormalization effects, would yield results that agree with mine.

[6]The Thirring model was proposed by W. Thirring, Ann. Phys. (N.Y.) 3, 91 (1958). A formal operator solution was found by V. Glaser, Nuovo Cimento 9, 990 (1958). A more rigorous solution was found by K. Johnson, Nuovo Cimento 20, 773 (1961). The model was studied in more detail by C. Sommerfield, Ann. Phys. (N.Y.)

26, 1 (1963), and by B. Klaiber, in *Lectures in Theoretical Physics,* lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1967, edited by A. Barut and W. Brittin (Gordon and Breach, New York, 1968), Vol. X, part A.

[7]This normalization of the current is that recommended by Sommerfield (Ref. 6). Johnson (Ref. 6) uses a different normalization, which leads to a different definition of the coupling constant. If we denote Johnson's coupling constant by g', then

$$g' = 2\pi g/(2\pi + g).$$

The allowed range of g' is $2\pi > g' > -2\pi$.

[8]See the discussion of A. Wightman, in *High Energy Electromagnetic Interactions and Field Theory,* edited by M. Lêvy (Gordon and Breach, New York, 1967).

[9]S. Coleman, Commun. Math. Phys. 31, 259 (1973).

[10]A very similar transformation of fermions into bosons takes place in two-dimensional quantum electrodynamics of massless fermions (the Schwinger model). This aspect of the model has recently been clarified and emphasized by A. Casher, J. Kogut, and L. Susskind, Phys. Rev. D 10, 732 (1974). I am indebted to Leonard Susskind and Curtis Callan for discussions of this phenomenon.

[11]I thank R. Jackiw for discussions of this point.

[12]I do not mean to imply that these two methods are equivalent. They are not, any more than in atomic physics perturbation theory for the T matrix is equivalent to perturbation theory for the K matrix. However, in atomic physics, these are both small-coupling approximations, although one is a better small-coupling approximation than the other in certain circumstances (for example, resonance scattering). Likewise, there may be circumstances where one of the two methods mentioned above is a better small-coupling approximation than the other, although we do not have enough insight into either method at the moment to know what these circumstances are.

[13]B. Schroer, private communication.

[14]A. Luther and I. Peschel, Phys. Rev. B 9, 2911 (1974); A. Luther and V. Emery, Phys. Rev. Lett. 33, 598 (1974).

[15]S. Mandelstam, Phys. Rev. D (to be published).

[16]G. Dell'Antonio, Y. Frishman, and D. Zwanziger, Phys. Rev. D 6, 988 (1972).

[17]J. Fröhlich, private communication.